GEOMETRY ON NODAL CURVES II: CYCLE MAP AND INTERSECTION CALCULUS

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Abstract. We study the relative Hilbert scheme of a family of nodal (or smooth) curves via its (birational) *cycle map*, going to the relative symmetric product. We show the cycle map is the blowing up of the discriminant locus, which consists of cycles with multiple points. We derive an intersection calculus for Chern classes of tautological bundles on the relative Hilbert scheme, which has applications to enumerative geometry.

Consider a family of curves given by a flat projective morphism

$$\pi : X \to B$$

over an irreducible (and usually projective) base, with fibres

$$X_b = \pi^{-1}(b), b \in B$$

which are irreducible nonsingular for the generic b and at worst nodal for every b. Many questions in the classical projective and enumerative geometry of this family can be naturally phrased, and in a formal sense solved (see for instance [R]), in the context of the *relative Hilbert scheme*

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B),$$

which parametrizes length-m subschemes of X contained in fibres of π, and the natural *tautological vector bundles* that live on $X_B^{[m]}$. Typically, the questions include ones involving relative multiple points and multisecants in the family, and the formal solutions involve Chern numbers of the tautological bundles. Thus, turning these formal solutions into meaningful ones requires computing the Chern numbers in question.

This paper is a contribution to the study, both qualitative and enumerative, of the relative Hilbert scheme of a family of modal curves as above. We provide the

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following:
– a structure theorem for the cycle (or 'Hilb-to-Chow') map

$$\mathfrak{c}_m : X_B^{[m]} \to X_B^{(m)},$$

where $X_B^{(m)}$ is the relative symmetric product, showing that $\mathfrak{c}_m$ is equivalent to the blowing up of the *discriminant locus*

$$D^m \subset X_B^{(m)},$$

which parametrizes nonreduced cycles;
– when X is a smooth surface, an intersection calculus for certain 'tautological classes' allowing computation of the Chern numbers of the tautological bundles on $X_B^{[m]}$.

To be precise, this calculus, which is based on the structure theorem, actually takes place on the (full) flag-Hilbert scheme $W^m(X/B)$, parametrizing length-m flags of subschemes of fibres of X/B, whose basic theory was developed in [R]. Nonethless, the Chern numbers computed in this calculus are the same, up to an evident factor, as those on $X_B^{[m]}$. Using the calculus, it is possible to compute explicitly the expressions given in [R] for various multiple-point and multisecant cycles. The advantage of using $W^m(X/B)$ over $X_B^{[m]}$ is that the tautological classes are expressed as polynomials in *divisor* classes $\Gamma^{\lceil i \rceil}, i = 2, ..., m$, corresponding to certain diagonal loci, together with the classes coming from X itself. This allows us to work in the ring T^m generated by these classes, a ring that we call the *tautological ring* on $W^m(X/B)$. Working in T^m, one is effectively working with divisor classes– in fact, T^m contains explicit expressions for the *Chern roots* of the tautological bundles, which are convenient in computations. Thus, passage to $W^m(X/B)$ and its tautological ring may be viewed as a version of the familiar 'splitting principle'.

What our calculus does is, essentially, to compute the operator of multiplication by $\Gamma^{\lceil m \rceil}$ on T^m. To be precise, our method effectively yields a set of additive generators of T^m, together with rules for expressing the product of a generator with $\Gamma^{\lceil m \rceil}$ as linear combination of generators. Given the inductive structure in m of the T^m, this completely determines the ring structure on T^m, albeit with an apparent ambiguity if (and only if) our generators are linearly dependent. It seems reasonable to conjecture that our generators are in fact linearly independent, but we do not prove this. In any event, our calculus is certainly sufficient to compute the top-degree products, which are those with enumerative significance.

Note that if X is a smooth surface, there is a natural closed embedding

$$j_\pi^{[m]} : X_B^{[m]} \subset X^{[m]}$$

of the relative Hilbert scheme in the full Hilbert scheme of X, which is a smooth projective $2m$-fold. There is a large literature on Hilbert schemes of smooth surfaces and their cohomology and intersection theory, due to Ellingsrud-Strømme, Göttsche, Nakajima, Lehn and others, see [EG, L, LS, N] and references therein. In particular, Lehn [L] gives a formula for the Chern classes of the tautological bundles on the full Hilbert scheme $X^{[m]}$, from which one can derive a formula for the analogous classes on $X_B^{[m]}$ if X is a smooth surface, but this does not, to our

knowledge, yield Chern numbers (besides the top one) on $X^{[m]}$, much less $X_B^{[m]}$ (the two sets of numbers are of course different). Going from Chern *classes* to Chern *numbers* it a matter of working out the top-degree multiplicative structure, i.e. the intersection calculus. When X is a surface with trivial canonical bundle, Lehn and Sorger [LS] have given a rather involved description of the mutiplicative structure on the cohomology of $X^{[m]}$ in all degrees, not just the top one. While products on $X^{[m]}$ and $X_B^{[m]}$ are compatible $j_\pi^{[m]}$, it's not clear how to compute intersection products, especially intersection *numbers* on $X_B^{[m]}$ from products on $X^{[m]}$, even in case X has trivial canonical bundle. Indeed some of our additive generators directly involve the *fibre nodes* of the family X/B and do not appear to come from classes on $X^{[m]}$. In any event, the computing the relative cohomology of the pair $(X_B^{[m]}, X^{[m]})$ is an interesting problem that at the moment seems out of reach.

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0. Preliminaries

We define a combinatorial function that will be important in computations to follow. Denote by Q the closed 1st quadrant in the real (x, y) plane, considered as an additive cone. We will consider unbounded Q-invariant closed subsets $R \subset Q$ with the property that the boundary of R relative to Q consists of a finite number of finite horizontal and vertical segments with integral endpoints (the boundary of R in $\mathbb{R}^2$ will then consist of this plus two semi-infinite intervals, one on each axis). We call such R a *special infinite polygon*. The closure of the complement

$$S = R^c := \overline{Q \setminus R} \subset Q$$

has finite (integer) area and will be called a *special finite polygon*; in fact the area of S coincides with the number of integral points in S that are Q-interior, i.e. not in R; these are precisely the integer points (a, b) such that $[a, a+1] \times [b, b+1] \subset S$. Fixing a natural number m, the basic special finite polygon associated to m is

$$S_m = \bigcup_{i=1}^{m} [0, \binom{m-i+1}{2}] \times [0, \binom{i+1}{2}].$$

It has area

$$\alpha_m = \sum_{i=1}^{m-1} i \binom{m+1-i}{2} = 3\binom{m}{4} + 3\binom{m}{3} + m - 1$$

and associated special infinite polygon denoted R_m. Now for each integer $j = 1, ..., m-1$ we define a special infinite polygon $R_{m,j}$ as follows. Set

$$P_j = (-j, m+1-j),$$

$$R_{m,j} = (R_m \cup (R_m + P_j) \cup [0, \infty) \times [j, \infty)) \cap Q$$

(where $R_m + P_j$ denotes the translate of R_m by P_j in $\mathbb{R}^2$). Then let $S_{m,j} = R^c_{m,j}$,

$$\beta_{m,j} = \text{area}(S_{m,j}),$$

$$\beta_m = \sum_{j=1}^{m-1} \beta_{m,j}.$$

It is easy to see that

$$\beta_{m,1} = \binom{m}{2}, \beta_{m,j} = \beta_{m,m-j}$$

but otherwise we don't know a closed-form formula for these numbers in general. A few small values are

$$\beta_{2,1} = \beta_2 = 1$$

$$\vec{\beta}_3 = (3,3), \beta_3 = 6$$

$$\vec{\beta}_4 = (6,8,6), \beta_4 = 20$$

$$\vec{\beta}_5 = (10,15,15,10), \beta_5 = 50$$

$$\vec{\beta}_6 = (15,24,27,24,15), \beta_6 = 105.$$

For an interpretation for these numbers see §1.6 below.

1. The cycle map as blowup

1.1 Set-up. Let

$$\pi : X \to B \tag{1.1.1}$$

be a family of nodal (or smooth) curves with X, B smooth. Let $X^m_B, X^{(m)}_B$, respectively, denote the mth Cartesian and symmetric fibre products of X relative to B. Thus, there is a natural map

$$\omega_m : X^m_B \to X^{(m)}_B \tag{1.1.2}$$

which realizes its target as the quotient of its source under the permutation action of the symmetric group $\mathfrak{S}_n$. Let

$$\text{Hilb}_m(X/B) = X^{[m]}_B$$

denote the relative Hilbert scheme paramerizing length-m subschemes of fibres of π, and

$$\mathfrak{c} = \mathfrak{c}_m : X^{[m]}_B \to X^{(m)}_B \tag{1.1.3}$$

the natural *cycle map* (cf.[A]). Let $D^m \subset X^{(m)}_B$ denote the discriminant locus or 'big diagonal', consisting of cycles supported on $< m$ points (endowed with the reduced scheme structure). Clearly, D^m is a prime Weil divisor on $X^{(m)}_B$, birational to $X \times_B \text{Sym}^{m-2}(X/B)$, though it is less clear what the defining equations of D^m on $X^{(m)}_B$ are near singular points. The purpose of this section is to prove

Theorem 1. *The cycle map*

$$\mathfrak{c}_m : X_B^{[m]} \to X_B^{(m)}$$

is the blow-up of $D^m \subset X_B^{(m)}$.

1.2 Preliminary reductions. To begin with, we reduce the Theorem to a local statement over a neighborhood of a 1-point cycle $mp \in X_B^{(m)}$ where $p \in X$ is a node of $\pi^{-1}(\pi(p))$. Set

$$\Gamma^{(m)} = \mathfrak{c}_m^{-1}(D^m) \subset X_B^{[m]}. \tag{1.2.1}$$

It was shown in [R], and will be reviewed below, that $\mathfrak{c}_m$ is a small birational map (with fibres of dimension ≤ 1), and that $X_B^{[m]}$ is smooth. Consequently $\Gamma^{(m)}$ is an integral, automatically Cartier, divisor, and therefore $\mathfrak{c}$ factors through a map $\mathfrak{c}'$ to the blow-up $B_{D^m}(X_B^{(m)})$, and it suffices to show that $\mathfrak{c}'$ is an isomorphism, which can be checked locally.

Next, let $X^o \subseteq X$ denote the open subset consisting of regular points of π, i.e. points $x \in X$ where π is smooth (submersive) or equivalently, such that x is a smooth point of $\pi^{-1}(\pi(x))$. Note that the open subset $\mathrm{Sym}^m(X^o/B) \subseteq X_B^{(m)}$ is smooth and

$$\mathfrak{c}_m : \mathfrak{c}_m^{-1}(\mathrm{Sym}^m(X^o/B)) \to \mathrm{Sym}^m(X^o/B)$$

is an isomorphism. Therefore it will suffice to show $\mathfrak{c}_m$ is equivalent to the blowing-up of D^m locally near any cycle $Z \in X_B^{(m)}$ whose support meets the locus $X^\sigma \subset X$ of singular points of π (i.e. singular points of fibres). Writing

$$Z = \sum_{i=1}^k m_i p_i$$

with $m_i > 0, p_i$ distinct, we have a cartesian diagram

$$\begin{array}{ccc} \prod_{i=1}^k {}_B\, X_B^{[m_i]} & \xrightarrow{\prod \mathfrak{c}_{m_i}} & \prod_{i=1}^k {}_B\, X_B^{(m_i)} \\ e_1 \uparrow & & \uparrow d_1 \\ H & \to & S \\ e \downarrow & & \downarrow d \\ X_B^{[m]} & \xrightarrow{\mathfrak{c}_m} & X_B^{(m)} \end{array} \tag{1.2.2}$$

Where H is the natural inclusion correspondence on Hilbert schemes:

$$H = \{(\zeta_1, ..., \zeta_k, \zeta) \in \prod_{i=1}^k {}_B\, X_B^{[m_i]} \times X_B^{[m]} : \zeta_i \subseteq \zeta, i = 1, ..., k\},$$

and similarly for S.

Note that the right vertical arrows d, d_1 are isomorphisms between some neighborhoods U of Z and U' of $(m_1p_1, ..., m_kp_k)$ and the left vertical arrows e, e_1 are

isomorphisms between $\mathfrak{c}_m^{-1}(U)$ and $(\prod \mathfrak{c}_{m_i})^{-1}(U')$. Now by definition, the blow-up of $X_B^{(m)}$ in D^m is the Proj of the graded algebra

$$A(\mathcal{I}_{D^m}) = \bigoplus_{n=0}^{\infty} \mathcal{I}_{D^m}^n.$$

Note that

$$d^{-1}(D^m) = \sum p_i^{-1}(D^{m_i})$$

and moreover,

$$d^*(\mathcal{I}_{D^m}) = \bigotimes{}_B \; p_i^*(\mathcal{I}_{D^{m_i}})$$

where we use p_i generically to denote an ith coordinate projection. Therefore,

$$A(\mathcal{I}_{D^m}) \simeq \bigotimes{}_B \; p_i^* A(\mathcal{I}_{D^{m_i}})$$

as graded algebras, compatibly with the isomorphism

$$\mathcal{O}_{\prod\limits_{i=1}^{k}{}_B \; \mathrm{Sym}^{m_i}(X/B)} \simeq \bigotimes_{i=1}^{k}{}_B \; \mathcal{O}_{\mathrm{Sym}^{m_i}(X/B)}.$$

Now it is a general fact that Proj is compatible with tensor product of graded algebras, in the sense that

$$\mathrm{Proj}(\bigotimes{}_B \; A_i) \simeq \prod{}_B \; \mathrm{Proj}(A_i).$$

Consequently (1.2.2) induces another cartesian diagram with unramified vertical arrows

$$\begin{array}{ccc} \prod\limits_{i=1}^{k}{}_B \; X_B^{[m_i]} & \xrightarrow{\prod c'_{m_i}} & \prod\limits_{i=1}^{k}{}_B \; B_{D^{m_i}} X_B^{(m_i)} \\ \downarrow & & \downarrow \\ X_B^{[m]} & \xrightarrow{c'_m} & B_{D^m} X_B^{(m)}. \end{array} \tag{1.2.3}$$

To prove c'_m is an isomorphism, it suffices to prove that so is c'_{m_i} for each i. The upshot of this is that it suffices to prove $c = \mathfrak{c}_m$ is equivalent to the blow-up of $X_B^{(m)}$ in D^m, locally over a neighborhood of a cycle of the form mp where $p \in X$ is a singular point of π.

1.3 A local model. Fixing such a point p, we have coordinates on an affine neighborhood U of p in X so that π is given on U by

$$t = xy.$$

Then the relative cartesian product X_B^m, as subscheme of $X^m \times B$, is given by

$$x_1 y_1 = ... = x_m y_m = t. \tag{1.3.1}$$

Let $\sigma_i^x, \sigma_i^y, i = 0, ..., m$ denote the elementary symmetric functions in $x_1, ..., x_m$ and in $y_1, ..., y_m$, respectively, where we set $\sigma_0 = 1$. Put together with the projection to B, they yield a map

$$\sigma : \mathrm{Sym}^m(U/B) \to \mathbb{A}_B^{2m} = \mathbb{A}^{2m} \times B \tag{1.3.2}$$

$$\sigma = ((-1)^m \sigma_m^x, ..., -\sigma_1^x, (-1)^m \sigma_m^y, ..., -\sigma_1^y, \pi^{(m)})$$

where $\pi^{(m)} : X_B^{(m)} \to B$ is the structure map.

Lemma 2. *σ is an embedding locally near mp.*

proof. It suffices to prove this formally, i.e. to show that $\sigma_i^x, \sigma_j^y, i, j = 1, ..., m$ generate topologically the completion $\hat{\mathfrak{m}}$ of the maximal ideal of mp in $X_B^{(m)}$. To this end it suffices to show that any $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j is a polynomial in the σ_i^x, σ_j^y and t. Let us denote by R the averaging or symmetrization operator with respect to the permutation action of $\mathfrak{S}_m$, i.e.

$$R(f) = \frac{1}{m!} \sum_{g \in \mathfrak{S}_m} g^*(f).$$

Then it suffices to show that the elements $R(x^I y^J)$, where x^I (resp. y^J) range over all monomials in $x_1, ..., x_m$ (resp. $y_1, ..., y_m$) are polynomials in the σ_i^x, σ_j^y and t. Now the relation (1.3.1) defining X_B^m easily implies that

$$R(x^I y^J) - R(x^I) R(y^J) = tF$$

where F is an $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j of bidegree $(|I|-1, |J|-1)$, hence a linear combination of elements of the form $R(x^{I'} y^{J'}), |I'| = |I| - 1, |J'| = |J| - 1$. By induction, F is a polynomial in the σ_i^x, σ_j^y and clearly so is $R(x^I)R(y^J)$. Hence so is $R(x^I y^J)$ and we are done. $\square$

Now let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates u_i, v_i on the i-th copy. Let $\tilde{C} \subset C_1 \times ... \times C_{m-1} \times B$ be the subscheme defined by

(1.3.3) $$v_1 u_2 = t u_1 v_2, ..., v_{m-2} u_{m-1} = t u_{m-2} v_{m-1}.$$

Thus $\tilde{C}$ is a reduced complete intersection of divisors of type $(1, 1, 0, ..., 0), (0, 1, 1, 0, ..., 0)$,..., $(0, ..., 0, 1, 1)$ and it is easy to check that the fibre of $\tilde{C}$ over $0 \in B$ is

$$\tilde{C}_0 = \bigcup_{i=1}^{m} \tilde{C}_i,$$

where

$$\tilde{C}_i = [1, 0] \times ... \times [1, 0] \times C_i \times [0, 1] \times ... \times [0, 1]$$

and that in a neighborhood of $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over B. We may embed $\tilde{C}$ in $\mathbb{P}^{m-1} \times B$, relatively over B using the mutihomogenous monomials

$$Z_i = u_1 \cdots u_{i-1} v_i \cdots v_{m-1}, i = 1, ..., m.$$

These satisfy the relations

(1.3.4) $$Z_i Z_j = t Z_{i+1} Z_{j-1}, i < j - 1$$

so they embed $\tilde{C}$ as a family of rational normal curves $\tilde{C}_t \subset \mathbb{P}^{m-1}, t \neq 0$ specializing to $\tilde{C}_0$, which is embedded as a nondegenerate, connected $(m-1)$-chain of lines.

Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, ..., a_{m-1}, d_0, ..., d_{m-1}$ and let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$a_0 u_1 = t v_1, d_0 v_{m-1} = t u_{m-1}$$

$$a_1u_1 = d_{m-1}v_1, ..., a_{m-1}u_{m-1} = d_1v_{m-1}. \tag{1.3.5}$$

Set $L_i = p_{C_i}^*\mathcal{O}(1)$. Then consider the subscheme of $Y = \tilde{H} \times_B U$ defined by the equations

$$F_0 := x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0 \in \Gamma(Y, \mathcal{O}_Y)$$

$$F_1 := u_1x^{m-1} + u_1a_{m-1}x^{m-2} + ... + u_1a_2x + u_1a_1 + v_1y \in \Gamma(Y, L_1)$$

...

$$F_i := u_ix^{m-i}+u_ia_{m-1}x^{m-i-1}+...+u_ia_{i+1}x+u_ia_i+v_id_{m-i+1}y+...+v_id_{m-1}y^{i-1}+v_iy^i$$

$$\in \Gamma(Y, L_i) \tag{1.3.6}$$

...

$$F_m := d_0 + d_1y_1 + ... + d_{m-1}y^{m-1} + y^m \in \Gamma(Y, \mathcal{O}_Y).$$

The following statement summarizes results from [R1]

Theorem 3. *(i) $\tilde{H}$ is smooth and irreducible.*

(ii) The ideal sheaf $\mathcal{I}$ generated by $F_0, ..., F_m$ defines a subscheme of $\tilde{H} \times_B X$ that is flat of length m over $\tilde{H}$

(iii)The classifying map

$$\Phi = \Phi_{\mathcal{I}} : \tilde{H} \to Hilb_m(U/B)$$

is an isomorphism.

proof. The smoothness of $\tilde{H}$ is clear from the defining equations equations and also follows from smoothness of $\mathrm{Hilb}_m(U/B)$ once (ii) and (iii) are proven. To that end consider the point $q_i, i = 1, ..., m$, on the special fibre of $\tilde{H}$ over $\mathbb{A}_B^{2m}$ with coordinates

$$v_j = 0,\ \forall j < i; u_j = 0,\ \forall j \geq i.$$

Then q_i has an affine neighborhood U_i in $\tilde{H}$ defined by

$$U_i = \{u_j = 1,\ \forall j < i;\ v_j = 1,\ \forall j \geq i\}, \tag{1.3.7}$$

and these $U_i, i = 1, ..., m$ cover a neighborhood of the special fibre of $\tilde{H}$. Now the generators F_i admit the following relations:

$$u_{i-1}F_j = u_jx^{i-1-j}F_{i-1},\ 0 \leq j < i-1;\ v_iF_j = v_jy^{j-i}F_i,\ m \geq j > i$$

where we set $u_i = v_i = 1$ for $i = 0, m$. Hence $\mathcal{I}$ is generated there by F_{i-1}, F_i and assertions (ii), (iii) follow directly from Theorems 1,2 and 3 of [R1]. □

Remark 3.1. For future reference, we note that over U_i, a co-basis for the universal ideal $\mathcal{I}$ (i.e. a basis for $\mathcal{O}/\mathcal{I}$) is given by $1, ..., x^{m-i}, y, ..., y^{i-1}$. In view of the definition of the F_i (1.3.6), this is immediate from the fact just noted that, over U_i, the ideal $\mathcal{I}$ is generated by F_{i-1}, F_i, plus the fact that on U_i we have $u_{i-1} = v_i = 1$. □

Remark 1.3.2. For integers $\alpha, \beta \leq m$, consider the locus $X_B^{(m)}(\alpha, \beta)$ of cycles containing $\alpha p + y' + y$" where p is a node and y', y" are general cycles of degree β (resp. $m - \alpha - \beta$) on the two (smooth) components of the special fibre. Then it is easy to see that the general fibre of $\mathfrak{c}_m$ over $X_B^{(m)}(\alpha)$ coincides with $\bigcup\limits_{i=m-\beta-\alpha+1}^{m-\beta-1} C_i^m$, which may be naturally identified with $C^\alpha = \bigcup\limits_{j=1}^{\alpha-1} C_j^\alpha$.

1.4 Reverse engineering. In light of Theorem 3, we identify a neighborhood H_m of the special fibre in $\tilde{H}$ with a neighborhood of the punctual Hilbert scheme (i.e. $\mathfrak{c}_m^{-1}(mp)$) in $X_B^{[m]}$, and note that the projection $H_m \to \mathbb{A}^{2m} \times B$ coincides generically, hence everywhere, with $\sigma \circ \mathfrak{c}_m$. Hence H_m may be viewed as the subscheme of $\mathrm{Sym}^m(U/B) \times_B \tilde{C}$ defined by the equations

$$\sigma_m^x u_1 = tv_1,$$

$$\sigma_{m-1}^x u_1 = \sigma_1^y v_1, ..., \sigma_1^x u_{m-1} = \sigma_{m-1}^y v_{m-1}, \tag{1.4.1}$$

$$tu_{m-1} = \sigma_m^y v_{m-1}$$

Alternatively, H_m may be defined as the subscheme of $\mathrm{Sym}^m(U/B) \times \mathbb{P}^{m-1} \times B$ defined by the relations (1.3.3), which define $\tilde{C}$, together with

$$\sigma_{m-j}^y Z_i = t^{m-j-i} \sigma_j^x Z_{i+1}, \quad i = 1, ..., m-1, j = 0, ..., m-1; \tag{1.4.2}$$

$$\sigma_{m-j}^x Z_i = t^{m-j-i} \sigma_j^y Z_{i-1}, \quad i = 2, ..., m, j = 0, ..., m-1. \tag{1.4.3}$$

Our task now is effectively to 'reverse-engineer' an ideal in the σ's whose syzigies are given by (1.4.2-1.4.3). To this end, it is convenient to introduce order in the coordinates. Thus let $OH_m = H_m \times_{\mathrm{Sym}^m(U/B)} U_B^m$, so we have a cartesian diagram

$$\begin{array}{ccc} OH_m & \xrightarrow{\varpi_m} & H_m \\ o\mathfrak{c}_m \downarrow & & \downarrow \mathfrak{c}_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

and its global analogue

$$\begin{array}{ccc} X_B^{\lceil m \rceil} & \xrightarrow{\varpi_m} & X_B^{[m]} \\ o\mathfrak{c}_m \downarrow & & \downarrow \mathfrak{c}_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array} \tag{1.4.4}$$

Note that $X_B^{(m)}$ is normal, Cohen-Macaulay and Q-Gorenstein: this follows from the fact that it is a quotient by $\mathfrak{S}_m$ of X_B^m, which is a locally complete intersection with singular locus of codimension ≥ 2 (in fact, > 2, since X is smooth). Alternatively, normality of $X_B^{(m)}$ follows from the fact that H_m is smooth and the fibres of $\mathfrak{c}_m : H_m \to X_B^{(m)}$ are connected (being products of connected chains of rational curves). Note that ω_m is simply ramified generically over D^m and we have

$$\omega_m^*(D^m) = 2OD^m$$

where

$$OD^m = \sum_{i<j} D_{i,j}^m$$

where $D_{i,j}^m = p_{i,j}^{-1}(OD^2)$ is the locus of points whose ith and jth components coincide. To prove $\mathfrak{c}_m$ is equivalent to the blowing-up of D^m it will suffice to prove

that $o\mathfrak{c}_m$ is equivalent to the blowing-up of $2OD^m = \omega_m^*(D^m)$ which in turn is equivalent to the blowing-up of OD^m. The advantage of working with OD^m rather than its unordered analogue is that at least some of its equations are easy to write down: let

$$v_x^m = \prod_{1\leq i<j\leq m} (x_i - x_j),$$

and likewise for v_y^m. As is well known, v_x^m is the determinant of the Van der Monde matrix

$$V_x^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{bmatrix}.$$

Also set

$$\tilde{U}_i = \varpi_m^{-1}(U_i),$$

where U_i is as in (1.3.7), being a neighborhood of q_i on H_m. Then in U_1, the universal ideal $\mathcal{I}$ is defined by

$$F_0, \quad F_1 = y + (\text{function of } x)$$

and consequently the length-m scheme corresponding to $\mathcal{I}$ maps isomorphically to its projection to the x-axis. Therefore over $\tilde{U}_1 = \varpi_m^{-1}(U_0)$, where F_0 splits as $\prod(x - x_i)$, the equation of OD^m is simply given by

$$G_1 = v_x^m.$$

Similarly, the equation of OD^m in $\tilde{U}_m$ is given by

$$G_m = v_y^m.$$

New let

$$\Xi : OH_m \to \mathbb{P}^{m-1}$$

be the morphism corresponding to $[Z_1, ..., Z_m]$, and set $L = \Xi^*(\mathcal{O}(1))$. Note that $\tilde{U}_i$ coincides with the open set where $Z_i \neq 0$, so Z_i generates L over $\tilde{U}_i$. Let

$$O\Gamma^{(m)} = o\mathfrak{c}_m^{-1}(OD^m).$$

This is a $1/2-$Cartier divisor because $2O\Gamma^{(m)} = \varpi_m^{-1}(\Gamma^{(m)})$ and $\Gamma^{(m)}$ is Cartier, H_m being smooth. In any case, the ideal $\mathcal{O}(-O\Gamma^{(m)})$ is a divisorial sheaf (reflexive of rank 1). Our aim is to construct an isomorphism

$$\gamma : \mathcal{O}(-O\Gamma^{(m)}) \to L. \tag{1.4.5}$$

Since $L = \Xi^*(\mathcal{O}(1))$ and OH_m is a subscheme of $X_B^m \times \mathbb{P}^{m-1}$, this isomorphism would clearly imply Theorem 1. To construct γ, it suffices to specify it on each $\tilde{U}_i$.

1.5 Mixed Van der Mondes and conclusion of proof. A clue as to how this might be done comes from the relations (1.4.2-1.4.3). Thus, set

$$G_i = \frac{(\sigma^y_m)^{i-1}}{t^{(i-1)(m-i/2)}} v^m_x = \frac{(\sigma^y_m)^{i-1}}{t^{(i-1)(m-i/2)}} G_1, \quad i = 2, ..., m. \tag{1.5.1}$$

Thus,

$$G_2 = \frac{\sigma^y_m}{t^{m-1}} G_1, G_3 = \frac{\sigma^y_m}{t^{m-2}} G_2, ..., G_{i+1} = \frac{\sigma^y_m}{t^{m-i}} G_i, i = 1, ..., m-1.$$

An elementary calculation shows that if we denote by V^m_i the 'mixed Van der Monde' matrix

$$V^m_i = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-i} & \dots & x_m^{m-i} \\ y_1 & \dots & y_m \\ \vdots & & \vdots \\ y_1^{i-1} & \dots & y_m^{i-1} \end{bmatrix} \tag{1.5.2}$$

then we have

$$G_i = \pm \det(V^m_i). \tag{1.5.3}$$

In particular, G_m as given in (1.5.1) coincides with v^m_y. I claim that G_i generates $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_i$. This is clearly true where $t \neq 0$ and it remains to check it along the special fibre $OH_{m,0}$ of OH_m over B. Note that $OH_{m,0}$ is a sum of components of the form

$$\Theta_I = \mathrm{Zeros}(x_i, i \notin I, y_i, i \in I), I \subseteq \{1, ..., m\},$$

none of which is contained in the singular locus of OH_m. Set

$$\Theta_i = \bigcup_{|I|=i} \Theta_I.$$

Note that

$$\tilde{C}_i \times 0 \subset \Theta_i, i = 1, ..., m-1$$

and therefore

$$\tilde{U}_i \cap \Theta_j = \emptyset, j \neq i-1, i.$$

Note that y_i vanishes to order 1 (resp. 0) on Θ_I whenever $i \in I$ (resp. $i \notin I$). Similarly, $x_i - x_j$ vanishes to order 1 (resp. 0) on Θ_I whenever both $i, j \in I^c$ (resp. not both $i, j \in I^c$). From this, an elementary calculation shows that the vanishing order of G_j on every component Θ of Θ_k is

$$\mathrm{ord}_\Theta(G_j) = (k-j)^2 + (k-j). \tag{1.5.4}$$

We may unambiguously denote this number by $\mathrm{ord}_{\Theta_k}(G_j)$. Since this order is nonnegative for all k, j, it follows firstly that the rational function G_j has no poles,

hence is in fact regular on X_B^m near mp (recall that X_B^m is normal); of course, regularity of G_j is also immediate from (1.5.3). Secondly, since this order is zero for $k = j, j-1$, and Θ_j, Θ_{j-1} contain all the components of $OH_{m,0}$ meeting $\tilde{U}_j$, it follows that in $\tilde{U}_j$, G_j has no zeros besides $O\Gamma^{(m)} \cap \tilde{U}_j$, so G_j is a generator of $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_j$.

Now since Z_j is a generator of L over $\tilde{U}_j$, we can define our isomorphism γ over $\tilde{U}_j$ simply by specifying that

$$\gamma(G_j) = Z_j \text{ on } \tilde{U}_j.$$

Now to check that these maps are compatible, it suffices to check that

$$G_j/G_k = Z_j/Z_k$$

as rational functions (in fact, units over $\tilde{U}_j \cap \tilde{U}_k$). But the ratios Z_j/Z_k are determined by the relations (1.4.2-1.4.3), while G_j/G_k can be computed from (1.5.3), and it is trivial to check that these agree. This completes the proof of Theorem 1. □

Corollary 4. *The ideal of OD^m is generated, locally near p^m, by $G_1, ..., G_m$.*

proof. We have

$$\mathcal{I}_{OD^m} = o\mathfrak{c}_{m*}(\mathcal{I}_{O\Gamma^{(m)}}) = o\mathfrak{c}_{m*}(L)$$

is generated by the images of $Z_1, ..., Z_m$, i.e. by $G_1, ..., G_m$.

As a further consequence, we can determine the ideal of the discriminant locus D^m itself: let δ_m^x denote the discriminant of F_0, which, as is well known [L], is a polynomial in the σ_i^x such that

$$\delta_m^x = G_1^2.$$

Set

$$\eta_{i,j} = \frac{(\sigma_m^y)^{i+j-2}}{t^{(i-1)(m-i)+(j-1)(m-j)}})\delta_x^m. \tag{1.5.5}$$

Corollary 5. *The ideal of D^m is generated, locally near mp, by $\eta_{i,j}, i, j = 1, ..., m$.*

proof. This follows from the fact that ϖ_m is flat and that

$$\varpi_m^*(\eta_{i,j}) = G_i G_j, i, j = 1, ..., m$$

generate the ideal of $2OD^m = \varpi_m^*(D^m)$.

Note that $\mathfrak{c}_m^*(D^m)$ is a Cartier divisor on $X_B^{[m]}$ (that, of course, is just the universal property of blowing up) but its ideal, that is, $\mathcal{O}(-\mathfrak{c}_m^*(D^m))$, is isomorphic in terms of our local model $\tilde{H}$ to $\mathcal{O}(2)$ (i.e. the pullback of $\mathcal{O}(2)$ from $\mathbb{P}^{m-1}$). This suggests that $\mathcal{O}(-\mathfrak{c}_m^*(D^m))$ is divisible by 2 as line bundle on $X_B^{[m]}$, as the following result indeed shows. First some notation. For a prime divisor A on X, denote by $[m]_*(A)$ the prime divisor on $X_B^{[m]}$ consisting of schemes whose support meets A. This operation is easily seen to be additive, hence can be extended to arbitrary, not necessarily effective, divisors and thence to line bundles.

Corollary 6. *Set*

$$\mathcal{O}_{X_B^{[m]}}(1) = \omega_{X_B^{[m]}} \otimes [m]_*(\omega_X^{-1}) \tag{1.5.6}$$

Then

$$\mathcal{O}_{X_B^{[m]}}(-\mathfrak{c}_m^*(D^m)) \simeq \mathcal{O}_{X_B^{[m]}}(2) \tag{1.5.7}$$

and

$$\mathcal{O}_{X_B^{\lceil m \rceil}}(-o\mathfrak{c}_m^*(OD^m)) \simeq \varpi_m^* \mathcal{O}_{X_B^{[m]}}(1). \tag{1.5.8}$$

proof. The Riemann-Hurwitz formula shows that the isomorphism (1.5.7) is valid on the open subset of $X_B^{[m]}$ consisting of schemes disjoint from the locus of fibre nodes of π. Since this open is big (has complement of codimension > 1), the iso holds on all of $X_B^{[m]}$. A similar argument establishes (1.5.8) $\square$

In practice, it is convenient to view (1.5.6) as a formula for $\omega_{X_B^{[m]}}$, with the understanding that $\mathcal{O}_{X_B^{[m]}}(1)$ coincides in our local model with the $\mathcal{O}(1)$ from the $\mathbb{P}^{m-1}$ factor, and that it pulls back over $X_B^{\lceil m \rceil} = X_B^{[m]} \times_{X_B^{(m)}} X_B^m$ to the $\mathcal{O}(1)$ associated to the blow up of the 'half discriminant' OD^m. We will also use the notation

$$\mathcal{O}(\Gamma^{(m)}) = \mathcal{O}_{X_B^{[m]}}(-1), \Gamma^{\lceil m \rceil} = \varpi_m^*(\Gamma^{(m)})$$

with the understanding that $\Gamma^{(m)}$ is Cartier, not necessarily effective, but $2\Gamma^{(m)}$ and $\Gamma^{\lceil m \rceil}$ are effective.

1.6 The small diagonal. The next Corollary will be crucial for the intersection calculus developed in the next section. It determines the restriction of the line associated to $\Gamma^{(m)}$, i.e. $\mathcal{O}_{X_B^{[m]}}(1)$, on the small diagonal. Thus let $\Gamma_{(m)} \subset X_B^{[m]}$ be the small diagonal, which parametrizes schemes with 1-point support, and which is the pullback of the small diagonal

$$D_{(m)} \simeq X \subset X_B^{(m)}.$$

The restriction of the cycle map yields a birational morphism

$$\mathfrak{c}_m : \Gamma_{(m)} \to X$$

which is an isomorphism except over the set of fibre nodes $\mathrm{sing}(\pi)$. Let

$$J_m^\sigma \subset \mathcal{O}_X$$

be the ideal sheaf whose stalk at each fibre node is of type J_m as in §0.

Corollary 7. *Via* $\mathfrak{c}_m$, $\Gamma_{(m)}$ *is equivalent to the blow-up of* J^σ_m*. If* $\mathcal{O}_{\Gamma_{(m)}}(1)$ *denotes the canonical blowup polarization, we have*

$$\mathcal{O}_{\Gamma_{(m)}}(-\Gamma^{(m)}) = \omega_{X/B}^{\otimes\binom{m}{2}} \otimes \mathcal{O}_{\Gamma_{(m)}}(1). \tag{1.6.1}$$

proof. We may work with the ordered versions of these objects, locally over a neighborhood of a point $p^m \in X^m_B$ where p is a fibre node. There the ideal of OD^m is generated by $G_1, ..., G_m$ and G_1 has the Van der Monde form v^m_x, while the other G_i are given by (1.5.1). We try to restrict the ideal of OD^m on the small diagonal $OD_{(m)}$. To this end, note that

$$(x_i - x_j)|_{OD_{(m)}} = dx = x\frac{dx}{x}$$

and $\eta = \frac{dx}{x}$ is a local generator of $\omega_{X/B}$. Therefore

$$G_1|_{OD_{(m)}} = x^{\binom{m}{2}}\eta^{\binom{m}{2}}.$$

From (1.5.1) we then deduce

$$G_i|_{\Gamma_{(m)}} = x^{\binom{m+i-1}{2}}y^{\binom{i}{2}}\eta^{\binom{m}{2}}, i = 1, ..., m. \tag{1.6.2}$$

Since $G_1, ..., G_m$ generate the ideal I_{OD^m}, it follows that

$$I_{OD^m} \otimes \mathcal{O}_{OD_{(m)}} \simeq J^\sigma_m \otimes \omega^{\binom{m}{2}}.$$

Consequently, we also have

$$I_{D^m} \otimes \mathcal{O}_{D_{(m)}} \simeq J^\sigma_m \otimes \omega^{\binom{m}{2}}.$$

Then pulling back to $X^{[m]}_B$ we get (1.6.1). □

Now working locally at a point p (which may be assumed a fibre node, though this is irrelevant for what follows), consider the blowup $c : \Gamma \to X$ of a punctual ideal of type J_m, and let e_m be the exceptional divisor, defined by

$$\mathcal{O}_\Gamma(1) := \mathcal{O}_\Gamma(-e_m) = c^*J_m$$

(pullback of ideals). Clearly the support of e_m is $C^m = \bigcup_{i=1}^{m-1} C^m_i$, so we can write

$$e_m = \sum_{i-1}^{m-1} b_{m,i}C^m_i$$

and we have

$$-e^2_m = \deg(\mathcal{O}(1).e_m) = \sum_{i=1}^{m-1} b_{m,i} =: b_m.$$

Now the general point on C_i^m corresponds to an ideal $(x^{m-i+1}+ay^i), a\in\mathbb{C}^*$ and the rational function x^{m-i+1}/y^i restricts to a coordinate on C_i^m. It follows that if $A_i\subset X$ is the curve with equation $f_i=x^{m-i+1}-ay^i, a\in\mathbb{C}^*$, then its proper transform $\tilde{A}_i$ meets C^m transversely in the unique point $q\in C_i^m$ with coordinate a, so that

$$\tilde{A}_i.e_m=b_{m,i}.$$

Thus, setting $J_{m,i}=J_m+(f_i)$ we get following characterization of $b_{m,i}$:

$$b_{m,i}=\ell(\mathcal{O}_X/J_{m,i}). \tag{1.6.3}$$

To compute this, we start by noting that a cobasis B_m for J_m, i.e. a basis for $\mathcal{O}_X/J_m$ is given by the monomials x^ay^b where (a,b) is an interior point of the polygon S_m as in §0; equivalently, the square with bottom left corner (a,b) lies in R_m. Then a cobasis $B_{m,i}$ for $J_{m,i}$ can be obtained by starting with B_m and eliminating

- all monomials x^ay^b with $b\geq i$;
- for any j with $\binom{j}{2}\geq i$, all monomials that are multiples of $x^{\binom{m+1-j}{2}+m+1-i}y^{\binom{j}{2}-i}$;

the latter of course comes from the relation

$$x^{\binom{m+1-j}{2}}y^{\binom{j}{2}}\equiv 0 \mod J_m.$$

Graphically, this cobasis corresponds exactly to the polygon $R_{m,i}$ in §0, hence

$$b_{m,i}=\beta_{m,i}, b_m=\beta_m; \tag{1.6.4}$$

in particular

Corollary 8. *With the above notations, we have globally*

$$e_m^2=-\sigma\beta_m, \tag{1.6.5}$$

$$\int\limits_{\Gamma_{(m)}}(\Gamma^{(m)})^2=-\sigma\beta_m+\binom{m}{2}^2\omega_{X/B}^2. \tag{1.6.6}$$

Remark 8.1. The components $C_i^m, i=1,...,m-1$ of e_m are special cases of the *node scrolls*, to be introduced in §2.3 below; the general node scroll is a $\mathbb{P}^1$ bundle whose fibre is an $e_{m,i}$. The coefficients $\beta_{m,i}$ play an essential role in the intersection calculus to be developed in §2.

For the remainder of the paper, we set

$$\omega=\omega_{X/B} \tag{1.6.7}$$

(viewed mainly as divisor class).

2. The tautological ring

We continue with the notations and assumptions of §1 and assume additionally that X is a smooth surface and B is a smooth curve. Our aim is to study the intersection theory associated to the tautological quotient bundle over the relative Hilbert scheme $X_B^{[m]}$. Thus let

$$\Lambda_m = \mathrm{Spec}(\mathcal{O}_{X_B^{[m]}\times_B X}/\mathcal{I}_m)$$

be the universal length-m subscheme, and for any vector bundle E on X, set

$$\lambda_m(E) = p_{1*}(p_2^*(E)\otimes\mathcal{O}_{\Lambda_m}).$$

By flatness of Λ_m over $X_B^{[m]}$, $\lambda_m(E)$ is clearly locally free of rank $m.\mathrm{rk}(E)$ on $X_B^{[m]}$. Our plan is first to review a formula for (essentially) the Chern classes of $\lambda_m(E)$, called *tautological classes*. More percisely, we will shift our situs operandi from the Hilbert scheme to its flag analogue. As a result, we are able to express the (pullback of the) tautological classes in terms of certain 'diagonal' *divisorial* classes (of Chow degree 1), essentially just the class $\Gamma^{(m)}$ defined above and its lower-degree analogues. We then work out the products of tautological classes in the Chow (or cohomology) ring of $X_B^{[m]}$, including especially the top-degree products, i.e. the Chern numbers of $\lambda_m(E)$, which might be called the *tautological numbers*. In the applications of the Hilbert scheme to classical enumerative geometry, it is these numbers that are required. We proceed, in fact, by giving a set of *additive generators* for the ring generated by the tautological classes $c_i(\lambda_m(E))$, and giving a calculus for expressing the product of one of these generators by $\Gamma^{(m)}$ as a linear combination of other generators. This is sufficient to compute all tautological numbers.

2.1 Divisorial multiplicative genrators. The total Chern class $c(\lambda_m(E))$ has been computed elsewhere in similar contexts: [L] in the case of the (full) Hilbert scheme of a smooth surface, [R] in the context of the relative flag-Hilbert scheme of a family of nodal curves over a base of any dimension. Our main goal is to compute the Chern numbers of $\lambda_m(E)$, and we note that Chern numbers, i.e. 'top' degree polynomials in the Chern classes, have a different meaning for the $(m+1)$-dimensional $X_B^{[m]}$ than for the $2m-$dimensional Hilbert scheme of X. Accordingly Lehn's formula [L] will be of no direct use to us. Rather, we will use the approach of [R] which has the advantage of yielding degree-1 (i.e. divisorial) multiplicative generators for the canonical ring, albeit at the cost of passing from the Hilbert scheme to its flag analogue. We now proceed to recall the required statement from [R].

Let

$$W^m = W^m(X/B) \xrightarrow{\pi^{(m)}} B$$

denote the relative flag-Hilbert scheme of X/B, parametrizing flags of subschemes

$$z. = (z_1 < ... < z_m)$$

where z_i has length i and z_m is contained in some fibre of X/B. Let

$$w^m : W^m \to X_B^{[m]}, w^{\lceil m\rceil} : W^m \to X_B^{\lceil m\rceil}$$

be the canonical (forgetful) maps. Let

$$p_i : W^m \to X$$

be the canonical map sending a flag $z.$ to the 1-point support of z_i/z_{i-1} and

$$p^m = \prod p_i : W^m \to X^m_B$$

their (fibred) product, which might be called the 'ordered cycle map'. W^m carries Cartier divisors

$$\Delta^{(i)} = \sum_{j=1}^{i-1} \Delta^i_j$$

with each Δ^i_j a prime Weil divisor defined generically by $p_i(z.) = p_j(z.)$ (thus $\Delta^{(1)} = 0$). We have

$$w^*_m(\Gamma^{(m)}) = (w^{\lceil m\rceil})^*(\Gamma^{\lceil m\rceil}) = \sum_{i=2}^{m} \Delta^{(i)}. \tag{2.1.1}$$

The formula of [R], Cor. 3.2 states that for any vector bundle E, we have

$$c(w^*_m\lambda_m(E)) = \prod_{i=1}^{m} c(p^*_i(E)(-\Delta^{(i)})) \tag{2.1.2}$$

In particular, if $E = L$ is a line bundle, we have

$$c(w^*_m\lambda_m(L)) = \prod_{i=1}^{m}(1 + [L^{(i)}] - [\Delta^{(i)}]) \tag{2.1.3}$$

where

$$L^{(i)} = p^*_i(L).$$

In [R2] we showed that (2.1.3) can be used to derive a more 'explicit' sum-of-products formula for $c(\lambda_m(L))$ on $X^{[m]}_B$ which, when X is a smooth surface, agrees with the restriction of a formula for the analogous bundles on $\mathrm{Hilb}_m(X)$ due to Lehn [L]. For the purpose of computing Chern numbers, obviously either W^m or $X^{[m]}_B$ could be used since the set of numbers they yield differ by a factor of $m!$. We will work in the former context, where the simple product formula (2.1.2) holds. Note that this formula has the added advantage of yielding directly the the *Chern roots* of $w^*\lambda_m(L)$, which are useful in computations.

In view of (2.1.2), we call the subring $T^m = T^m(X/B)$ of the $\mathbb{Q}$-Chow ring of W^m generated by the $\Delta^{(i)}$ and the $p^*_i(A^{\cdot}(X)), i = 2, ..., m$ the *tautological ring* of W^m. In view of (2.1.1), we may replace the generators $\Delta^{(i)}, i = 2, ..., m$ by $\Gamma^{(i)}$ or $\Gamma^{\lceil i\rceil}, i = 2, ..., m$ which are more convenient (e.g. $\Gamma^{(i)}$ lives on $X^{[i]}_B$). By their very definition, the various T^m's form a chain

$$T^2 \to ... \to T^{m-1} \to T^m.$$

Assuming X is a surface, so that $\dim W^m = m+1$, we will give a method, inductive in m, to express an arbitrary nonzero monomial M in T^m in terms of certain additive generators (to be specified below), assuming the analogous result in T^{m-1} is known. We may assume that M is a monomial in $\Gamma^{\lceil 2\rceil}, ..., \Gamma^{\lceil m\rceil}$, hence expressible in the form

$$M = M'(\Gamma^{\lceil m\rceil})^r$$

with $M' \in T^{m-1}$. By induction on m, we may assume M' is already expressed as a linear combination of the additive generators. Therefore, we may as well assume M' is itself one of the additive generators in T^{m-1}. Then, using induction on r, it will suffice to show how to express the product of an additive generator in W^m by $\Gamma^{\lceil m\rceil}$ as a linear combination of additive generators.

Now our additive generators for the tautological ring come in three flavors: the *diagonals*, analogous to Nakajima's creation operators; the *node scrolls*, which are certain $\mathbb{P}^1-$bundles parametrizing schemes whose support contains some fibre nodes; and the *node sections*, which are certain cross-sections of node scrolls. We first introduce the diagonal classes.

2.2 Diagonal classes. Note that for any pair of distinct pairs $(i < j) \neq (i' < j')$, the intersection

$$\Delta_i^j \cap \Delta_{i'}^{j'}$$

is a well-defined codimension-2 cycle on W^m, because Δ_i^j and $\Delta_{i'}^{j'}$ are Cartier at the generic point of the intersection. Similarly, for any index-set

$$I = (i_1 < ... < i_k) \subset [1, m]$$

and any $c \in H^\cdot(X)$, we have a well-defined cycle class that we call a *connected diagonal monomial*

$$q_I[c] = c^{(i_1)}\Delta_{i_1}^{i_2}\Delta_{i_2}^{i_3}...\Delta_{i_{k-1}}^{i_k} = c^{(i_1)}\Delta_I. \tag{2.2.1}$$

When necessary to indicate the dependence on m we'll sometimes write this as $q_I^{(m)}[c]$. When I is a singleton $\{i\}$, (2.2.1) reads

$$q_i[c] = c^{(i)}.$$

$q_I[c]$ is an ordered analogue of Nakajima's creation operator $q_{|I|}[c]$ (cf. [N, EG]). Likewise, for any partition $(I.) = (I_1, ..., I_h)$, i.e. collection $I_1, ..., I_h \subset [1, m]$ of pairwise disjoint subsets or 'blocks', with associated classes $c_1, ..., c_h$, we have a well-defined (disconnected, if $h > 1$) *diagonal monomial*

$$q_{(I.)}[c.] = q_{I_1}[c_1] \cdots q_{I_h}[c_h].$$

We view $(I.)$ as a sort of disconnected set with $I_1, ..., I_h$ its connected components, and $(c.)$ as a locally constant $H^\cdot(X)$-valued function on $(I.)$.. Note that $q_{(I.)}[c.]$ is supported on

$$\Delta_{(I.)} = \Delta_{I_1} \cap \cdots \cap \Delta_{I_h} \sim q_{I_1}[1] \cdots q_{I_h}[1]$$

which maps under the ordered cycle map to the appropriate diagonal locus $OD^m_{(I.)}$. It is obvious from (2.1.3) that the Chern classes of $w_m^*\lambda_m(L)$ are linear combinations

of diagonal monomials. The coefficients are worked out in [R2], and are consistent with Lehn's formula in [L]. We call the group generated by the diagonal monomials $q_{(I.)}[(c.)]$ the group of *diagonal classes.*

It is worth noting that the diagonal classes $q_I[c] = q_I^{(m)}[c]$ behave simply with respect to push-forward and pullback under the natural map

$$\gamma^{m,m-1} : W^m \to W^{m-1}.$$

First,

$$I \subset [1, m-1] \Rightarrow (\gamma^{m,m-1})^* q_I^{(m-1)}[c] = q_I^{(m)}[c] \tag{2.2.2}$$

(consequently, it is safe to omit the superscript from $q_I^{(m)}[c]$); next,

$$m \in I, |I| > 1 \Rightarrow \gamma_*^{m.m-1} q_I[c] = q_{I\cap[1,m-1]}^{(m-1)}[c]; \tag{2.2.3}$$

$$I = (m) \Rightarrow \gamma_*^{m,m-1}(q_{(m)}[c]) = \gamma_*^{m,m-1}(c^{(m)}) = (\pi^{(m-1)})^* \pi_*^{(m)}(c) \tag{2.2.4}$$

(if c is of Chow degree 1 (cohomological degree 2), this is just $\deg_\pi(c) 1_{W^{m-1}}$ where $\deg_\pi(c)$ is the fibre degree).

$$\gamma_*^{m.m-1} q_I[c] = 0, m \notin I. \tag{2.2.5}$$

Analogous formulae hold also for diagonal monomials $q_{(I.)}[c.]$. By the projection formula, it follows in particular that $\gamma_*^{m.m-1} q_{(I.)}[c.]$ is a diagonal monomial in T^{m-1}. Using this inductively, we see that that for any 0-dimensional (degree-$(m+1)$) diagonal monomial $q_{(I.)}[c.]$, we can easily compute the number

$$\int_{W^m} q_{(I.)}[c.].$$

Unfortunately, the group generated by the diagonal classes is not closed under multiplication by Δ^i or Γ^i classes; achieving closure requires introduction of node scroll and node section classes.

2.3 Node scrolls. Consider a partition

$$I_1 \coprod I_2 \coprod J_1 ... \coprod J_a \coprod K_1 ... \coprod K_b \subseteq [1, m]$$

such that

$$|I_1|, |I_2| > 0$$

and that I_1 contains the smallest I_1 elements of $I_1 \cup I_2$ in terms of the usual ordering on $[1, m]$; thus I_1 is an 'initial segment' of $I_1 \cup I_2$. We will call

$$\Phi = (I_1 | I_2 : J | K)$$

a set of *partition data* with respect to m. More generally, if I_1 is not an initial segment of $I_1 \cup I_2$, we identify $(I_1|I_2 : J|K)$ with $(I_1'|I_2' : J|K)$ where I_1' is the

initial segment of $I_1 \cup I_2$ of cardinality $|I_1|$ and $I'_2 = (I_1 \cup I_2) \setminus I'_1$. The case where J or K is empty is included, and if both are empty we will write $\Phi = (I_1|I_2 :)$. We think of Φ as indexing some of the variables $x_1, y_1, ..., x_m, y_m$ where I_1, J (resp. I_2, K) refer to y (resp. x) variables.

Φ is said to be *full* if

$$\bigcup \Phi := I_1 \cup ... \cup K_b = [1, m].$$

A *filling* of Φ is a full set of partition data $\Phi' = (I_1|I_2 : J'|K')$ such that J' (resp. K') differs from J (resp. K) only by 1-element blocks. We write this as $\Phi \prec \Phi'$ Let $X_1, ..., X_\sigma$ be the singular fibres of π. Assume first that the singular fibre X_s is a union of two smooth components X'_s, X''_s meeting in a single point n_s, and fix $\Phi = (I_1|I_2 : J|K)$. We also let n'_s, n_s" denote the preimages of n_s on $X'_s, X"_s$, respectively. Set

$$X_s^\Phi = \prod_a p_{J_a}^{-1}(\Delta_{X"_s}) \times \prod_a p_{K_a}^{-1}(\Delta_{X'_s}) \times \prod_{i \in I_1 \cup I_2} p_i^{-1}(n_s) \tag{2.3.1}$$

where $\Delta_{X'_s} \subset (X'_s)^{K_a}$ etc. denotes the *small* diagonal. This depends on I_1, I_2 only via $I_1 \cup I_2$. Note that the irreducible components of X^Φ are precisely the $X^{\Phi'}$ where Φ' is a filling of Φ, and each of these is a smooth subvariety of X_B^m, isomorphic to $(X_s")^{\ell(J)}(X'_s)^{\ell(K)}$, where $\ell(J)$ denotes the number of blocks (or 'length') of the partition J. Fix such a filling Φ'. We have

$$\sigma_i^y|_{X_s^{\Phi'}} = 0, i > |J'| := \sum |J'_j|,$$

$$\sigma_i^x|_{X_s^{\Phi'}} = 0, i > |K'|.$$

In light of the relations (1.4.1), which hold in a local model of the Hilbert scheme near the 'origin' n_s^m, this implies that the generic fibre of p^m over $X_s^{\Phi'}$ in a neighborhood of the 'origin', described in terms of this model, has the form

$$\bigcup_{i=|J'|+1}^{m-|I'|-1} C_i^m. \tag{†}$$

Setting $r = m - |J'| - |K'| = |I_1| + |I_2|$, this same fibre can be identified, in terms of a local model of Hilb over a *generic* point of $X_s^{\Phi'}$, with

$$\bigcup_{i=1}^{r-1} C_i^r \tag{‡}$$

(cf. Remark 1.3.2). We denote by

$$F_s^{\Phi'} \subset X_B^{\lceil m \rceil} \tag{2.3.2}$$

the component of $w^{\lceil m \rceil}((p^m)^{-1}(X_s^{\Phi'}))$ with generic fibre $C^m_{|J'|+|I_1|}$ in the first identification (†) or $C^r_{|I_1|}$ in the second (‡). When there is no confusion, we may use the

same notation for the preimage of F_s^Φ in W^m, i.e. $(p^m)^{-1}(X_s^{\Phi'})$. This depends on I_1, I_2 only via $I_1 \cup I_2, |I_1|$ (recall that I_1 is an 'initial segment' of $I_1 \cup I_2$). Informally, we think of I_1 and J (resp. I_2 and K) as indexing y (resp. x) variables, where the I variables are localized at the origin and the J, K variables are free.

The natural map

$$p^{\Phi'} : F_s^{\Phi'} \to X_s^{\Phi'} \tag{2.3.3}$$

is a $\mathbb{P}^1$-bundle (this is of course only true of the model of F_s^Φ in $X_B^{\lceil m \rceil}$) . The locus $F_s^{\Phi'}$ is called the *node scroll* corresponding to the node s and the partition data Φ'. We also set

$$F_s^\Phi = \bigcup_{\Phi \prec \Phi' \text{ full}} F_s^{\Phi'}, \tag{2.3.4}$$

We now indicate the modifications needed to construct node scrolls for an irreducible 1-nodal fibre X_s. For a set of partition data $\Phi = (I_1|I_2 : J|K)$ we now insist that $K = \emptyset$ and that Φ be full. Let X'_s be the normalization of X_s, marked with the 2 node preimage n'_s, n_s". Set

$$X_s^\Phi = \prod_a p_{J_a}^{-1}(\Delta_{X'_s}) \times \prod_{i \in I_1} p_i^{-1}(n"_s) \times \prod_{i \in I_2} p_i^{-1}(n'_s) \subset (X'_s)^m,$$

which is the direct analogue of (2.3.1). Note that this locus has $2^{\ell(J)}$ natural 'origins', viz. the elements of

$$\prod_a p_{J_a}^{-1}\{(n'_s)^{J_a}, (n"_s)^{J_a}\} \times \prod_{i \in I_1} p_i^{-1}(n"_s) \times \prod_{i \in I_2} p_i^{-1}(n'_s)$$

where $(n'_s)^{J_a}$ is the diagonal point corresponding to n'_s etc. Let

$$n : X_s^\Phi \to X_s^m \subset X_B^m$$

be the natural map induced by normalization, and set

$$F_s^\Phi = (p^m)^{-1}(n(X_s^\Phi)).$$

We note that the restriction of p^m lifts to a $\mathbb{P}^1$-bundle projection

$$p^\Phi : F_s^\Phi \to X_s^\Phi.$$

Indeed, this may be checked locally analytically on X_s^Φ and there is clear from our local analytic model for the Hilbert scheme, in which the branches of X_s at the node already appear separated, and the target of the cycle map appears as the product of the symmetric products of the branches. Finally, set

$$F^\Phi = \sum_{s=1}^{\sigma} F_s^\Phi$$

(sum over all singular fibres, both reducible and irreducible).

A *node section* class is by definition a class of the form $-\Gamma^{\lceil m\rceil}.F_s^\Phi$. The group generated by the classes of node scrolls and node sections is called the group of *node classes*. This group and the operation of $\Gamma^{\lceil m\rceil}$ on it will be sudied at length in §2.5.

One obvious fact worth noting at the outset is that for $\Phi = (I_1|I_2 : J|K)$, if $i \in I_1 \cup I_2$ and $c \in H^*(X)$ is a class of positive degree (codimension), then

$$c^{(i)}.[F_s^\Phi] = 0, \forall s$$

(e.g. because c admits a representative disjoint from $\mathrm{sing}(\pi)$). It follows that

$$I \cap (I_1 \cup I_2) \neq \emptyset, \deg(c) > 0 \Rightarrow q_I[c].[F_s^\Phi] = 0 = q_I[c].(\Gamma^{\lceil m\rceil}.[F_s^\Phi]). \tag{2.3.5}$$

2.4 Cutting a diagonal class. Our aim in this subsection is to express the product of a diagonal class by $\Gamma^{\lceil m\rceil}$ as a linear combination of diagonal classes and node (scroll) classes, generalizing the results of §1.6. To this end, note that for any multi-index (1-block partition) I and any $i \in I$, the projection

$$g = p_i : \Delta_I \to X$$

is independent of $i \in I$ and thus Δ_I maps birationally, via the ordered cycle map, to

$$X \times_B (\prod_{j\notin I} {}_B X).$$

The generic fibre of the induced map $\Delta_I \to \prod\limits_{j\notin I} {}_B X$ is isomorphic to the 'small diagonal' $\Gamma_{(|I|)}$ which parametrized 1-point schemes. Recall that the intersection $\Gamma^{(m)}.\Gamma_{(m)} = \Gamma^{\lceil m\rceil}.\Gamma_{(m)}$ was computed in §1.6. A similar reasoning shows that $\Delta_I.\Gamma^{\lceil m\rceil}$ can be computed as the sum of the following terms

• $\sum \Delta_{(I:(a,b))}$, the sum being over all $a < b$ with both $a, b \notin I$, where $(I : (a, b))$ is the obvious 2-block partition;

• $\sum \Delta_{I\cup\{a,b\}}$, sum over all $a < b$ with $|I \cap \{a, b\}| = 1$, where $I \cup \{a, b\}$ is the obvious block;

• $q_I[\omega^{\binom{|I|}{2}}]$;

• $\sum\limits_{j=1}^{|I|-1} \beta_{|I|,j} F^{((i_1,\dots,i_j|i_{j+1},\dots,j_{|I|}:)}$.

In order to write this compactly, the following purely combinatorial gadget will appear frequently below. Let $J = \{j_1, j_2\}, j_1 \neq j_2$ be an index pair, and $(I.)$ a partition. A new partition $(I'.) = J \ltimes (I.)$ is obtained from $(I.)$ as follows.

- if $j_1 \in I_a, j_2 \in I_b$ for some a, b, remove I_a, I_b from $(I.)$ and inset $I_a \cup I_b$ (in other words, 'connect up' I_a and I_b, reducing the number of blocks (or 'connected components') by 1;

- if $j_1 \in I_a, j_2 \notin I_b, \forall b$, or vice versa, replace I_a by $I_a \cup J$;

- if $j_1, j_2 \notin I_a, \forall a$, insert J to $(I.)$ as a block (thus increasing by 1 the number of connected components).

- if $J \subset I_a$ for some a, $(I'.) = (I.)$.

With this notation, we can rewrite our formula for $\Gamma^{\lceil m\rceil}.\Delta_I$ as follows.

$$\Gamma^{\lceil m\rceil}.\Delta_I = \sum_{i<j} \Delta_{\{i,j\}\ltimes I} + \sum_{j=1}^{|I|-1} \beta_{|I|,j} F^{((i_1,\dots,i_j|i_{j+1},\dots,j_{|I|}:)} + \binom{|I|}{2} q_I[\omega]. \tag{2.4.1}$$

The extension of (2.4.1) to the case of (disconnected) diagonal monomials is straightforward. For notational economy it is convenient to denote the middle term in (2.4.1) by $F^{(I:)}$; we similarly have $F^{(I:J|K)}$ for partitions $(I:J|K)$. From this it is easy to see that more generally, we have

$$\Gamma^{\lceil m\rceil}.\Delta_{(I.)} = \sum_{i<j} \Delta_{\{i,j\}\ltimes(I.)} + \sum_{k} \sum_{J\cup K=I.\setminus I_k} F^{(I_k:J|K)} \tag{2.4.2}$$

$$+\sum_{k} q_{I_1}[1]\cdots q_{I_k}[\binom{|I_k|}{2}\omega]\cdots q_{I_h}[1]$$

where the last two sums may be restricted to those I_k such that $|I_k|\geq 2$, as the others yield 0 and, as always, for an irreducible singular fibre X_s the condition that $K=\emptyset$ in F^{Φ}_s remains in force. For instance, in terms of the generator G_1, the first term in (2.4.2) corresponds to the factors x_i-x_j of G_1 such that $\{i,j\}$ are not in the same block of $(I.)$; the second and 3rd terms come from the various k so that $\{i,j\}\subset I_k$.

It is a routine matter, albeit necessary, to extend (2.4.2) to a formula for diagonal monomials $\Gamma^{\lceil m\rceil}.q_{(I.)}[c.]$. To state this, we need yet some more notation. For any pluri-multi-index $(I.)=(I_1,...,I_h)$ and classes $c_1,...,c_h\in H^*(X)$ (or $A^*(X)$), let us denote the diagonal monomial $q_{I_1}[c_1]\cdots q_{I_h}[c_h]$ by $q_{(I.)}[(c.)]$. Here the pluri-class $(c.)$ should be viewed as a function from $(I.)$ to $H^*(X)$. Then for a distinct pair $J=\{j_1,j_2\}$, there is a natural way to modify $(c.)$ to define a pluri-class $J\ltimes(c.)$ on $J\ltimes(I.)$:

- in case I_a and I_b get connected up to form $I_a\cup I_b$, i.e. $j_1\in I_a, j_2\in I_b$ or vice versa, the value of $J\ltimes(c.)$ on $I_a\cup I_b$ is $c_a._X c_b$;
- in case I_a gets replaced by $I_a\cup J$, the value of $J\ltimes(c.)$ on $I_a\cup J$ is (c_a);
- in case J is inserted to $(I.)$, define the value of $J\ltimes(c.)$ on J to equal $1\in H^*(X)$; all other values are carried over from $(c.)$ to $J\ltimes(c.)$ in the obvious way.

Also, if $(c.)$ is a pluri-class on $(I:J|K)$, define $F^{(I:J|K)}_s[(c.)]$ as follows.

$$F^{(I:J|K)}_s[(c.)]=0 \quad\text{if}\quad \deg c(I)>0;$$

$$F^{(I:J|K)}_s[(c.)] = F^{(I:J|K)}\prod c(J_a)^{(\min(J_a))}\prod c(K_a)^{(\min(K_a))} \quad\text{if}\quad c(I)=1. \tag{2.4.3}$$

These are called generalized node scroll classes, and we similarly have generalized node section classes. Note that (2.4.3) clearly vanishes if $c(J_a)$ or $c(K_a)$ is of degree >1. Also set

$$X^{(I:J|K)}_s[(c.)] = X^{(I:J|K)}_s\prod c(J_a)^{(\min(J_a))}\prod c(K_a)^{(\min(K_a))}.$$

Note that this is a 0-cycle precisely when

$$\ell(J)+\ell(K)=\sum_a \deg(c(J_a))+\sum_a \deg(c(K_a))$$

where $\ell(J),\ell(K)$ denote the number of blocks in the partition (which coincides with the dimension of $X^{(I:J|K)}_s$); in other words, $X^{(I:J|K)}_s[(c.)]$ is a 0-cycle precisely when each $c(J_a),c(K_a)$ is of degree 1. In this case we have

$$\int_{W^m} X^{(I:J|K)}_s[(c.)] = \int_{X^{(I:J|K)}_s}\prod c(J_a)^{(\min(J_a))}\prod c(K_a)^{(\min(K_a))}$$

$$= \prod_a \deg_\pi(c(J_a)) \prod_a \deg_\pi(c(K_a))$$

With this notation, the extension of (2.4.2) reads

$$\Gamma^{\lceil m\rceil}.q_{(I.)}[c.] = \sum_{1\leq i<j\leq r} q_{\{i,j\}\ltimes(I.)}[\{i,j\}\ltimes(c.)] \tag{2.4.4}$$

$$+\sum_k \sum_{J\cup K=I.\setminus I_k} F^{(I:J|K)}[(c.)] + \sum_k q_{I_1}[c_1]\cdots q_{I_k}[\binom{|I_k|}{2}\omega.c_k]\cdots q_{I_h}[c_h]$$

We have thus shown that the product of any diagonal class with $\Gamma^{(m)}$ can be expressed in terms of diagonal classes and (generalized) node classes. We can now state the main result of this section:

Theorem 4. *Any element of the tautological ring T^m can be (computably) expressed as a linear combination of diagonals and generalized node classes.*

The plan is to prove by induction on m, so we may assume it holds for all $m' < m$. Note that the minimum dimension for a generalized node scroll class $F_s^\Phi[(c.)]$ (resp. generalized node section $-\Gamma^{\lceil m\rceil}.F_s^\Phi[(c.)]$) is 1 (resp. 0), both achieved when $X_s^\Phi.[(c.)]$ is 0-dimensional, so in view of the obvious fact, when $X_s^\Phi([(c.)]$ is a 0-cycle, that

$$\int_{W^m} -\Gamma^{\lceil m\rceil}.F_s^\Phi[(c.)] = \int_{X_B^m} X_s^\Phi([(c.)]) \tag{2.4.5}$$

(the latter being the degree of a 0-cycle) Theorem 4 allows us to compute $\int_{W^m} M$ for any top-degree element $M\in T^m$, as was our main goal.

2.5 Cutting a node class. It remains to analyze the product of a generalized node class with $\Gamma^{(m)}$ (i.e. with $\Gamma^{\lceil m\rceil}$). We will do this for ungeneralized node classes, as the extension to the case of generalized node classes is straightforward. To this end, we wish first to analyze the structure of a node scroll F_s^Φ with $\Phi = (I. : J|K)$ a full set of partition data. To be able to state formulae uniformly the reducible and irreducible singular fibres, it is convenient to set

$$\begin{aligned} K' \quad &= K, \text{ reducible case} \\ &= J, \text{ irreducible case} \end{aligned}$$

$$\begin{aligned} K" \quad &= K, \text{ reducible case} \\ &= \emptyset, \text{ irreducible case} \end{aligned}$$

As noted earlier, the natural map

$$p^\Phi : F_s^\Phi \to X_s^\Phi$$

exhibits F_s^Φ as a $\mathbb{P}^1$-bundle, and we wish to identify the corresponding vector bundle. Assume to simplify notation that $I_1 = [1,i], I_2 = [i+1,r]$. Recall that homogeneous coordinates on C_i^r are given by Z_i, Z_{i+1} which correspond to the mixed Van der Monde generators G_i, G_{i+1}; ditto for $C^m_{|J|+i}$. Consider the mixed Van der Monde

matrix $V^m_{|J|+i}$ whose determinant yields $G_{|J|+i}$. It has an $r\times r$ block submatrix based on the I-indexed rows and the columns, corresponding to $1, x, ..., x^{r-i}, y, ..., y^{i-1}$, whose determinant is equal to $G_{i,I}$, that is, the G_i expression in the variables $x_1, y_1, ..., x_r, y_r$. Note that this is globally defined along X^Φ_s. The determinant of the complementary submatrix, considered as function on X^Φ_s, is a 'shift' of another Van der Monde, equal to

$$(x^{K'})^{r-i}(y^J)^i \prod_{a<b\in\bigcup K'}(x_a - x_b)\prod_{a<b\in\bigcup J}(y_a - y_b), \tag{2.5.1}$$

where $x^{K'} = \prod\limits_{k\in\bigcup K'} x_k$ etc and, for X_s irreducible, x, y are local coordinates at the node preimages $n'_s, n"_s$, respectively. Note that in the irreducible nodal case, the last 2 factors in (2.5.1) define the same diagonal locus, the one near $(n'_s)^{K'}$, the other near $(n"_s)^J$. Now (2.5.1) is a generator of the invertible ideal

(2.5.2)

$${}'E^\Phi_s = \mathcal{O}(-(r-i)\sum_{a\in\bigcup K'} p^*_a n'_s - i\sum_{a\in\bigcup J} p^*_a n"_s - \sum_{a,b\in\bigcup K"} p^*_{a,b}(\Delta) - \sum_{a,b\in\bigcup J} p^*_{a,b}(\Delta)).$$

Other terms in the Laplace expansion of $G_{|J|+i}$ along the I columns have order $> \binom{r}{2} = \mathrm{ord}(G_{i,I})$ in the I variables. Analogous considerations for the second Van der Monde generator $G_{|J|+i+1}$ lead to the invertible ideal

(2.5.3)

$${}"E^\Phi_s = \mathcal{O}(-(r-i-1)\sum_{a\in\bigcup K'} p^*_a n'_s - (i+1)\sum_{a\in\bigcup J} p^*_a n"_s - \sum_{a,b\in\bigcup K"} p^*_{a,b}(\Delta) - \sum_{a,b\in\bigcup J} p^*_{a,b}(\Delta)).$$

Setting

$$E^\Phi_s =' E^\Phi_s \oplus " E^\Phi_s, \tag{2.5.4}$$

we conclude that, at least in a neighborhood of the 'origin' n^m_s, we have

$$F^\Phi_s \simeq \mathbb{P}(E^\Phi_s) \tag{2.5.5}$$

so that

$$\mathcal{O}(-\Gamma^m)|_{F^\Phi_s} = \mathcal{O}_{\mathbb{P}(E^\Phi_s)}(1). \tag{2.5.5'}$$

A similar argument shows that this isomorphism persists near 'less special' points on X^Φ_s, namely, expanding $G_{|J|+i}$ we again get, modulo higher-order terms, the same $G_{i,I}$ factor times another local generator of ${}'E^\Phi_s$ and likewise for $G_{|J|+i+1}$; so the isomorphism (2.5.5)-(2.5.5') holds globally. Note that

$$\mathbb{P}(E^\Phi_s) = \mathbb{P}(\mathcal{O}(-\sum_{a\in J} p^*_a n"_s) \oplus \mathcal{O}(-\sum_{a\in K'} p^*_a n'_s)) \tag{2.5.6}$$

but the latter bundle gives the 'wrong' $\mathcal{O}(1)$.

Next it is important to compare node classes on W^{m-1} and W^m. Let $\Phi = (I_1|I_2 : J|K)$ be full partition data with respect to $[1, m-1]$. In the reducible case, there are precisely two completions of Φ with respect to $[1, m]$, namely

$$\Phi' = (I_1|I_2 : J^+ = J\cup\{m\}|K), \Phi" = (I_1|I_2 : J|K^+ = K\cup\{m\}).$$

In the irreducible case, there is just Φ'. There is a natural sheaf inclusion

$$E_s^{\Phi'} \to p_{[1,m-1]}^* E_s^{\Phi}(-ip_m^*(n_s) - \sum_{a\in\bigcup J} p_{a,m}^*(\Delta)) \tag{2.5.7}$$

which drops rank by 1 with multiplicity 1 along $p_m^{-1}(n_s)$, identifying $F_s^{\Phi'}$ as an elementary modification of F_s^{Φ}, albeit with polarization

$$-\Gamma^{(m)}.F_s^{\Phi'} = (-\Gamma^{(m-1)} - (i+1)p_m^*(n_s) - \sum_{a\in\bigcup J} p_{a,m}^*(\Delta)).F_s^{\Phi} \tag{2.5.8}$$

(see Remark 2.5.1 below). In the reducible case, we have additionally

$$-\Gamma^{(m)}.F_s^{\Phi"} = (-\Gamma^{(m-1)} - (i+1)p_m^*(n_s) - \sum_{a\in\bigcup K} p_{a,m}^*(\Delta)).F_s^{\Phi}. \tag{2.5.8"}$$

In fact the model of $F_s^{\Phi'}$ on W^m is a blown-up $\mathbb{P}^1$-bundle which contracts on the one hand to $F_s^{\Phi} \subset X_B^{\lceil m-1\rceil}$ and on the other hand to $F_s^{\Phi'} \subset X_B^{\lceil m\rceil}$. Together with (2.5.8) and (2.5.8"), this implies that $(\gamma^{m,m-1})^*$ takes node classes on W^{m-1} to node classes on W^m. From this, it is obvious that the same is true for generalized node classes. Now to compute the Chern classes of E_s^{Φ}, note that

$$p_a^{-1}(n"_s) = [X_s^{(I_1\cup\{a\}|I_2:J\setminus\{a\}|K)}], a \in \bigcup J \tag{2.5.9}$$

$$p_a^{-1}(n'_s) = [X_s^{(I_1|I_2\cup\{a\}:J|K\setminus\{a\})}], a \in \bigcup K' \tag{2.5.10}$$

in the reducible case, and

$$p_a^{-1}(n'_s) = [X_s^{(I_1|I_2\cup\{a\}:J\setminus\{a\}|K)}], a \in \bigcup K' \tag{2.5.10'}$$

in the irreducible case;

$$p_{a,b}^*(\Delta) = \omega^{(a)} = \omega^{(\min(J_r))} = (2g(X_s") - 2)p_a^*(pt) \text{ if } \{a,b\} \subset J_r \tag{2.5.11}$$

$$p_{a,b}^*(\Delta) = \omega^{(a)} = \omega^{(\min(K'_r))} = (2g(X'_s) - 2)p_a^*(pt) \text{ if } \{a,b\} \subset K'_r. \tag{2.5.12}$$

$$p_{a,b}^*(\Delta) = [X_s^{(I.:(a,b)\ltimes J|K)}] \text{ if } \{a,b\} \not\subset J_r, \forall r, \{a,b\} \subset \bigcup J \tag{2.5.13}$$

$$p_{a,b}^*(\Delta) = [X_s^{(I.:J|(a,b)\ltimes K")}] \text{ if } \{a,b\} \not\subset K_r, \forall r, \{a,b\} \subset \bigcup K" \tag{2.5.14}$$

(here just (2.5.12) and (2.5.13) are operative in the irreducible case).
All these are codimension-1 classes on X_s^{Φ}, whose pullback via p^{Φ} are clearly generalized node classes. It follows that, in the irreducible case,
$c_1(E_s^{\Phi}) =$

$$-(2i+1)(p^{\Phi})^* \sum_{a\in J} [X_s^{(I_1\cup\{a\}|I_2:J\setminus\{a\}|K)}] - (2r-2i-1)(p^{\Phi})^* \sum_{a\in K} [X_s^{(I_1|I_2\cup\{a\}:J|K\setminus\{a\})}]$$

$$-2\sum_{a<b\in\bigcup K}(p^\Phi)^*[X_s^{(I.:J|(a,b)\ltimes K)}]-2(p^\Phi)^*[X_s^\Phi]\sum_r\binom{|K_r|}{2}\omega^{(\min(K_r))}$$

$$-2(p^\Phi)^*\sum_{a<b\in\bigcup J}[X_s^{(I.:(a,b)\ltimes J|K)}]-2(p^\Phi)^*[X_s^\Phi]\sum_r\binom{|J_r|}{2}\omega^{(\min(J_r))}$$

$$=-(2i+1)\sum_{a\in J}[F_s^{(I_1\cup\{a\}|I_2:J\setminus\{a\}|K)}]-(2r-2i-1)\sum_{a\in K}[F_s^{(I_1|I_2\cup\{a\}:J|K\setminus\{a\})}]$$

$$-2\sum_{a<b\in\bigcup K"}[F_s^{(I.:J|(a,b)\ltimes K")}]-2[F_s^\Phi]\sum_r\binom{|K_r|}{2}\omega^{(\min(K"_r))}$$

$$-2\sum_{a<b\in\bigcup J}[F_s^{(I.:(a,b)\ltimes J|K)}]-2[F_s^\Phi]\sum_r\binom{|J_r|}{2}\omega^{(\min(J_r))}; \tag{2.5.15}$$

in the irreducible case, the second summation is replaced by

$$\sum_{a\in K'}[F_s^{(I_1|I_2\cup\{a\}:K'\setminus\{a\}|\emptyset)}]$$

In the expression (2.5.15) the 1st 2 terms come from the 1st 2 terms in $'E,"E$; the 3rd and 4th terms come from the 3rd term in $'E,"E$ and correspond to the case where a,b are in different blocks (resp. the same block) of K; similarly for the 5th and 6th terms. In particular, $c_1(E_s^\Phi)$ is clearly a generalized node class. The computation of

$$c_2(E_s^\Phi)=c_1('E_s^\Phi)c_1("E_s^\Phi) \tag{2.5.16}$$

is straightforward: note that X_s^Φ is just a product of smooth curves and the classes being multiplied are standard ones. The following elementary facts may be used:

$$p_a^*(n_s)^2=0; \tag{2.5.17}$$

$$p_a^*(n_s)p_b^*(n_s)=p_{a,b}^*(pt)=p_{a,b}^*(\Delta)p_a^*(n_s), a\neq b$$

$$p_a^*(n"_s)p_b^*(n"_s)=X^{(I_1\cup\{a,b\}|I_2:J|K)}, \{a,b\}\subset\bigcup J$$

$$p_a^*(n'_s)p_b^*(n'_s)=X^{(I_1|I_2\cup\{a,b\}:J|K)}, \{a,b\}\subset\bigcup K' \tag{2.5.18}$$

$$p_a^*(n"_s)p_b^*(n'_s)=X^{(I_1\cup\{a\}|I_2\cup\{b\}:J|K)}, a\in\bigcup J, b\in\bigcup K';$$

$$p_{a,b}^*(\Delta)p_{c,d}^*(\Delta)=0; \tag{2.5.19}$$

if a,b,c,d are in the same block of J or K;

if a, b are in different blocks, then

$$p^*_{a,b}(\Delta)^2 = (2-2g)p^*_{a,b}(pt); \tag{2.5.20}$$

where $g = g(X")$ if $a, b \in \bigcup J$ or $g(X')$ if $a, b \in \bigcup K'$;
more generally, if a, b are in different blocks of J, then for all c, d,

$$p^*_{a,b}(\Delta)p^*_{c,d}(\Delta) = p^*_{c,d}(\Delta)|_{X^{(I_1|I_2:(a,b)\ltimes J|K)}}; \tag{2.5.21}$$

ditto if a, b are in different blocks of K.

Clearly $c_2(E^\Phi_s)$ is in the group of generalized node classes. Now Grothendieck's standard relation

$$c_2(E^\Phi_s(-1)) = 0$$

yields

$$(\Gamma^{(m)})^2.F^\Phi_s = -\Gamma^{(m)}.(p^\Phi)^*c_1(E^\Phi_s) - (p^\Phi)^*c_2(E^\Phi_s). \tag{2.5.22}$$

Therefore also

$$(\Gamma^{(m)})^2.F^\Phi_s[(c.)] = \tag{2.5.23}$$

$$-\Gamma^{(m)}.(p^\Phi)^*(c_1(E^\Phi_s).[X^\Phi_s[(c.)] - (p^\Phi)^*(c_2(E^\Phi_s).[X^\Phi_s[(c.)].$$

Applying this recursively, we see that the group of generalized node classes is closed under multiplication by $\Gamma^{(m)}$, which completes the proof of Theorem 4. $\square$

Remark 2.5.1. Let

$$u : E_1 \to E_0$$

be a map of rank-2 vector-bundles on a scheme X, which drops rank by 1 along a divisor Z, i.e. is locally of the form $\mathrm{diag}(1, z)$, where z is an equation of Z. Then u induces a rational map, known as an 'elementary modification'

$$\mathbb{P}(E_0) \dashrightarrow \mathbb{P}(E_1)$$

which is defined by a correspondence

$$\begin{array}{ccccc} & & Q & & \\ & \alpha \swarrow & & \searrow \beta & \\ \mathbb{P}(E_0) & & & & \mathbb{P}(E_1) \end{array}$$

where $Q \subset \mathbb{P}(E_0) \times_X \mathbb{P}(E_1)$ is the 0-locus of the natural map induced by u

$$p_2^*(M_{E_1}) \to p_1^*(\mathcal{O}_{\mathbb{P}(E_0)}(1))$$

where M_{E_1} is the tautological subbundle (which in this case coincides with $\mathcal{O}_{\mathbb{P}(E_1)}(-1)$ because E_1 has rank 2). Then

$$\beta^*(\mathcal{O}_{\mathbb{P}(E_1)}(1)) = \alpha^*(\mathcal{O}_{\mathbb{P}(E_0)}(1))(-Z). \tag{2.5.1.1}$$

Indeed (2.5.1.1) is obvious because by Q's definition there is a natural map induced by u, $\beta^*(\mathcal{O}_{\mathbb{P}(E_1)}(1)) \to \alpha^*(\mathcal{O}_{\mathbb{P}(E_0)}(1))$ and this has divisor of zeros precisely Z.

2.6 Example. With X/B as above (B a smooth curve), suppose $f : X \to \mathbb{P}^{2m-1}$ is a morphism. One, quite special, class of examples of this situation arises as what we call a *generic rational pencil*; that is, generally, the normalization of the family of rational curves of fixed degree d in $\mathbb{P}^r$ (so $r = 2m - 1$ here) that are incident to a generic collection $A_1, ...A_k$ of linear spaces, with

$$(r+1)d + r - 4 = \sum(\mathrm{codim}(A_i) - 1);$$

see [R3] and references therein, or [RA] for an 'executive summary'. Then one expects a finite number N_m of curves $f(X_b)$ to admit an m-secant $(m-2)$-plane, and this number can be evaluated as follows. Let $G = G(m-1, 2m)$ be the Grassmannian of $(m-2)$-planes in $\mathbb{P}^{2m-1}$, with rank-$(m+1)$ tautological subbundle S, and let $L = f^*\mathcal{O}(1)$. Then

$$m!N_m = \int\limits_{W^m \times G} c_{m(m+1)}(S^* \boxtimes w^*\lambda_m(L))$$

$$= \int\limits_{W^m \times G} c_{m+1}(S^*(L^{(1)}))c_{m+1}(S^*(L^{(2)} - \Delta^{(2)})) \cdots c_{m+1}(S^*(L^{(m)} - \Delta^{(m)}))$$

$$= \int\limits_{W^m \times G} \prod_{i=1}^{m}(\sum_{j=0}^{m+1} \binom{m+1}{j} c_{m+1-j}(S^*)(L^{(i)} - \Delta^{(i)})^j)$$

$$= \sum_{|(j.)|=m+1} \int\limits_{G} c_{m+1-j_1,...,m+1-j_m}(S^*) \int\limits_{W^m} (L^{(1)})^{j_1}(L^{(2)} - \Delta^{(2)})^{j_2}...(L^{(m)} - \Delta^{(m)})^{j_m}$$

where $c_{u,v,w} = c_u c_v c_w$. Note that only terms with $j_m > 0$ contribute. By the intersection calculus developed above, this number can be computed in terms of the characters $L^2, \deg_\pi(L), \omega^2, \sigma, \omega.L, \deg_\pi(\omega) = 2g - 2, g =$fibre genus; in the generic rational pencil case, all these characters can be computed by recursion on d.

Suppose now that $m = 3$, where the only relevant $(j.)$ are

$$(2,1,1),(1,1,2),(1,2,1),(1,0,3),(0,3,1),(0,2,2),(0,1,3),(0,0,4).$$

In each of these cases, it is easy to see that the G integral evaluates to 1. The W integrals may be evaluated by the calculus developed above. The relevant formulae are

(2.6.0) $$\int\limits_{W^3} u\Delta^{(3)} = 2\int\limits_{W^2} u, u \in T^2$$

(2.6.1) $$(\Delta^{(2)})^2 = (\Gamma^{\lceil 2 \rceil})^2 = F^{(12:)} + q_{12}[\omega]$$

(as usual we use $F^{(12:)}$ as short for $F^{(12:\emptyset|\emptyset)}$)

(2.6.2) $$\int_{W^2} L^{(i)}(\Delta^{(2)})^2 = L.\omega = 1/2 \int\limits_{W^3} L^{(i)}(\Delta^{(2)})^2\Delta^{(3)}, i = 1, 2$$

$$= 1/2 \int_{W^3} L^{(3)} (\Delta^{(2)})^2 \Delta^{(3)}, \tag{2.6.3}$$

$$\int_{W^2} L^{(i)} L^{(j)} \Delta^{(2)} = L^2 = 1/2 \int_{W^3} L^{(i)} L^{(j)} \Delta^{(2)} \Delta^{(3)}, (i,j) = (1,1), (1,2), (2,2) \tag{2.6.4}$$

$$= 1/2 \int_{W^3} L^{(i)} L^{(3)} \Delta^{(2)} \Delta^{(3)}, i = 1,2,3; \tag{2.6.5}$$

$$\int_{W^2} L^{(1)} (L^{(2)})^2 = \deg_\pi(L) L^2 = 1/2 \int_{W^3} (L^{(1)}) L^{(2)} L^{(3)} \Delta^{(3)} =$$

$$= \int_{W^3} (L^{(1)})^i (L^{(2)})^j (L^{(3)})^k \Delta^{(3)}, (i,j,k) = (1,0,2), (0,1,2) \tag{2.6.6}$$

$$\int_{W^2} (\Delta^{(2)})^3 = -\sigma + \omega^2 = 1/2 \int_{W^3} (\Delta^{(2)})^3 \Delta^{(3)} \tag{2.6.7}$$

$$(\Delta^{(3)})^2 = 2q_{123}[1] - q_{13}[\omega] - q_{23}[\omega] + F^{(13:)} + F^{(23:)} \tag{2.6.8}$$

where $F_s^{(i3:)} = \mathbb{P}(\mathcal{O}(-n_s))$ over $X'_s \coprod X"_s$, with the 'correct' $\mathcal{O}(1)$, i=1,2;

$$L^{(3)}.(\Delta^{(3)})^2 = 2q_{123}[L] - q_{13}[\omega.L] - q_{23}[\omega.L]$$

$$\begin{array}{rcl} \int_{W^3} L^{(3)} L^{(i)}.(\Delta^{(3)})^2 & = & 2L^2 - \deg_\pi(L) L.\omega, \quad i = 1,2 \\ & = & 2L^2, \quad i = 3 \end{array} \tag{2.6.9}$$

$$\int_{W^3} u(\Delta^{(3)})^2 = \int_{W^2} u(2\Delta^{(2)} - \omega^{(1)} - \omega^{(2)}), u = L^{(1)}\Delta^{(2)} = L^{(2)}\Delta^{(2)}, (\Delta^{(2)})^2$$

(we can ignore F terms because u is perpendicular to them by (2.3.5))

$$= \int_{W^2} L^{(1)} (2q_{12}[-\omega] - q_{12}[\omega] - q_{12}[\omega])$$

$$= -4L\omega, \quad \text{if } u = L^{(1)}\Delta^{(2)} = L^{(2)}\Delta^{(2)} \tag{2.6.10}$$

$$= \int_{W^2} (\Delta^{(2)})^2(2\Delta^{(2)} - \omega^{(1)} - \omega^{(2)}) = \int\limits_{W^2} 2(\Delta^{(2)})^3 + 2q_{12}[\omega^2]$$

$$= -2\sigma + 4\omega^2, \text{ if } u = (\Delta^{(2)})^2; \tag{2.6.11}$$

(2.6.12)

$$(\Delta^{(3)})^3 = \begin{array}{c} 2(\Gamma^{\lceil 3\rceil} - \Gamma^{\lceil 2\rceil})q_{123}[1] \\ -2q_{123}[\omega] \quad +q_{13}[\omega^2] + q_{23}[\omega^2] + (\Gamma^{\lceil 3\rceil} - \Gamma^{\lceil 2\rceil})(F^{(13:)} + F^{(23:)}) \end{array}$$

$$\int\limits_{W^3} L^{(i)}(\Delta^{(3)})^3 = 2L.\omega, i = 1, 2 \tag{2.6.13}$$

$$\int\limits_{W^3} \Delta^{(2)}(\Delta^{(3)})^3 = -6\sigma + 8\omega^2 \tag{2.6.14}$$

$$\int\limits_{W^3} (\Delta^{(3)})^4 = 2(-3\sigma + 4\omega^2) + 2.2.\omega^2 + \omega^2 + \omega^2 + 2(-2\sigma + 4\sigma)$$

$$= -2\sigma + 14\omega^2 \tag{2.6.15}$$

where we have used the facts

$$(\Gamma^{\lceil 3\rceil})^2.\Gamma_{(3)} = \Gamma^{\lceil 3\rceil}(F^{(123:)} + 3q_{123}[\omega]) = -6\sigma + 9\omega^2, (\Gamma^{\lceil 2\rceil})^2.\Gamma_{(3)} = -\sigma + \omega^2,$$

(both by §1.6, as $\beta_{3,1} = \beta_{3,2} = 3, \beta_{2,1} = 1$)

$$\Gamma^{\lceil 3\rceil}\Gamma^{\lceil 2\rceil}\Gamma_{(3)} = \frac{1}{2}(\Gamma^{\lceil 3\rceil})^2(\Gamma^{\lceil 2\rceil})^2 = \frac{1}{2}\int\limits_{W^3} (\Gamma^{\lceil 3\rceil})^2\gamma^{3,2*}(F^{(12:)} + q_{12}[-\omega])$$

$$= \frac{1}{2}\int\limits_{F^{(12:)}} (\Gamma^{\lceil 3\rceil})^2 + \frac{1}{2}\int\limits_{W^3} \Gamma^{(3)}(q_{12}[\omega^2] + 2q_{123}[-\omega])$$

$$= \frac{1}{2}\int\limits_{F^{(12:)}} (\Gamma^{\lceil 2\rceil} - 2.\text{fibre})^2 + \frac{1}{2}\int\limits_{W^3} 2q_{123}[\omega^2] + 2q_{123}[(-\omega).(-\omega)]$$

$$= -2\sigma + 3\omega^2$$

(for the last equality, note that $F^{(12:)}$ is a single point on W^2 so $(\Gamma^{\lceil 2\rceil})^2 = 0$ on $F^{(12:)}$ and likewise on its pullback on W^3);

$$(\Gamma^{\lceil 3\rceil})^2.F^{(i3:)} = -2\sigma, (\Gamma^{\lceil 2\rceil})^2.F^{(i3:)} = 0, \Gamma^{\lceil 3\rceil}\Gamma^{\lceil 2\rceil}.F^{(i3:)} = -2\sigma, i = 1, 2.$$

From all these, the evaluation of N_3 is routine.

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